

INVERSE GALOIS PROBLEM FOR SMALL SIMPLE GROUPS

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ABSTRACT. In this short note, we list all simple groups with cardinality at most a hundred million that are not known to occur as the Galois group of an extension of \mathbb{Q} .

Recall that the **Inverse Galois Problem (IGP)** asks whether every finite group G occurs as the Galois group of some Galois extension of \mathbb{Q} . We will focus on the fundamental case where G is simple.

Below we give a list of all non-abelian simple groups, up to isomorphism, with cardinality at most 10^8 . We have given a reference for those groups for which the IGP is known. Those groups without a reference, and in [blue](#), have not been realized as a Galois extension of \mathbb{Q} (at least as far as the author is aware¹). For notation and a statement of the classification of finite simple groups, see [\[CCN⁺85\]](#). We have excluded abelian simple groups since the IGP is known for them.

Disclaimer: I have only given the most convenient reference, and have not tried to find the first occurrence in the literature or assign credit. The goal of this note is not to summarize what has been done, but to determine which small cases of the IGP still remain open.²

- $\mathrm{PSL}_2(\mathbb{F}_p)$ with prime $2 \leq p \leq 577$ [\[Zyw12\]](#) (earlier results of Shih and Malle cover the cases $p \notin \{311, 479\}$)
- $\mathrm{PSL}_2(\mathbb{F}_{p^2})$ with prime $2 \leq p \leq 23$ [\[Shi04, DV00\]](#)
- $\mathrm{PSL}_2(\mathbb{F}_{2^3})$ [\[Mat87\]](#)
- $\mathrm{PSL}_2(\mathbb{F}_{2^n})$ with $4 \leq n \leq 8$ [\[Wie05\]](#)
- $\mathrm{PSL}_2(\mathbb{F}_{3^n})$ with $3 \leq n \leq 5$ [\[DFV22\]](#)
- $\mathrm{PSL}_2(\mathbb{F}_{5^3})$ [\[DFV22\]](#)
- $\mathrm{PSL}_2(\mathbb{F}_{7^3})$ [\[DFV22\]](#)
- \mathfrak{A}_7 [\[Ser08, §4.4\]](#)
- $\mathrm{PSL}_3(\mathbb{F}_3)$ [\[Rei99\]](#)
- $\mathrm{PSU}_3(\mathbb{F}_3)$ [\[Rei99\]](#)
- Mathieu group M_{11} [\[MM99, II §9\]](#)
- \mathfrak{A}_8 [\[Ser08, §4.4\]](#)
- $\mathrm{PSL}_3(\mathbb{F}_4)$ (cf. [§1](#) below)
- $\mathrm{PSU}_4(\mathbb{F}_2) \cong \mathrm{PSp}_4(\mathbb{F}_3)$ [\[Rei99\]](#)
- Suzuki group ${}^2B_2(8)$
- $\mathrm{PSU}_3(\mathbb{F}_4)$
- Mathieu group M_{12} [\[MM99, II §9\]](#)
- $\mathrm{PSU}_3(\mathbb{F}_5)$ [\[DFV22\]](#)
- Janko group J_1 [\[MM99, II §9\]](#)
- \mathfrak{A}_9 [\[Ser08, §4.4\]](#)
- $\mathrm{PSL}_3(\mathbb{F}_5)$ [\[Rei99\]](#)
- Mathieu group M_{22} [\[MM99, II §9\]](#)
- Janko group J_2 [\[MM99, II §9\]](#)
- $\mathrm{PSp}_4(\mathbb{F}_4)$ [\[Shi03\]](#)
- $\mathrm{Sp}_6(\mathbb{F}_2)$ [\[Rei99\]](#)
- \mathfrak{A}_{10} [\[Ser08, §4.4\]](#)
- $\mathrm{PSL}_3(\mathbb{F}_7)$ [\[DFV22\]](#)
- $\mathrm{PSU}_4(\mathbb{F}_3)$ [\[Shi03\]](#)
- $G_2(\mathbb{F}_3)$ [\[MM99, II §8.1\]](#)
- $\mathrm{PSp}_4(\mathbb{F}_5)$ [\[Rei99\]](#)
- $\mathrm{PSU}_3(\mathbb{F}_8)$

¹Any additional information would be greatly appreciated. Last updated August 2025.

²An earlier version of this list motivated [\[DFV22\]](#) that led to the IGP being addressed for 7 simple groups.

- $\text{PSU}_3(\mathbb{F}_7)$ [Rei99]
- $\text{PSL}_4(\mathbb{F}_3)$ [Rei99]
- $\text{PSL}_5(\mathbb{F}_2)$ [Kön17]
- Mathieu group M_{23}
- $\text{PSU}_5(\mathbb{F}_2)$ [Shi03]
- $\text{PSL}_3(\mathbb{F}_8)$
- Tits group ${}^2F_4(2)'$ [Shi03]
- \mathfrak{A}_{11} [Ser08, §4.4]
- Suzuki group ${}^2B_2(32)$
- $\text{PSL}_3(\mathbb{F}_9)$ [Shi03]
- $\text{PSU}_3(\mathbb{F}_9)$
- Higman–Sims group HS [MM99, II §9]
- Janko group J_3 [MM99, II §9]
- $\text{PSU}_3(\mathbb{F}_{11})$ [Mal90]

Finally, we summarize the open cases. The following is the list of simple groups G with cardinality at most 10^8 for which it is currently unknown if there is a Galois extension of \mathbb{Q} with Galois group G . (For each group, we also give its cardinality.)

- ${}^2B_2(8)$, 29120
- $\text{PSU}_3(\mathbb{F}_4)$, 62400
- $\text{PSU}_3(\mathbb{F}_8)$, 5515776
- M_{23} , 10200960
- $\text{PSL}_3(\mathbb{F}_8)$, 16482816
- ${}^2B_2(32)$, 32537600
- $\text{PSU}_3(\mathbb{F}_9)$, 42573600

1. REALIZATION OF $\text{PSL}_3(\mathbb{F}_4)$ AS A GALOIS GROUP

In this section, using the work of G. Malle, we give a polynomial in $\mathbb{Q}[x]$ whose splitting field over \mathbb{Q} has Galois group $\text{PSL}_3(\mathbb{F}_4)$. The group $\text{PSL}_3(\mathbb{F}_4)$ is simple with cardinality 20160.

Define the polynomials $q(x) := 11(39x^8 + 280x^7 + 4092x^6 - 4136x^5 + 110594x^4 - 146168x^3 + 984940x^2 - 734712x + 2668655)$ and $p(x) := 29x^7 - 165x^6 - 539x^5 + 363x^4 - 12705x^3 + 3993x^2 - 35937x - 49247$ in $\mathbb{Q}[x]$, and the polynomial

$$h(x, t) := (p(x)^2 q(x)(t^2 + 11)^{11} - p(t)^2 q(t)(x^2 + 11)^{11}) / (x - t) \in \mathbb{Q}(t)[x].$$

Malle proved that the Galois group of $h(x, t)$ over $\mathbb{Q}(t)$ is isomorphic to $H := \text{PSL}_3(\mathbb{F}_4) \cdot 2_2$ [Mal88, Theorem 3] (the notation for H is described in [CCN⁺85]). The group H has $\text{PSL}_3(\mathbb{F}_4)$ as a normal subgroup of index 2.

In the splitting field of $h(x, t)$ over $\mathbb{Q}(t)$, the degree 2 extension of $\mathbb{Q}(t)$ corresponding to the subgroup $\text{PSL}_3(\mathbb{F}_4)$ equals $\mathbb{Q}(t, T)$ where T satisfies $T^2 = q(t)$, cf. [Mal88, Prop. 2]. The (projective) curve defined by $T^2 = q(t)$ has genus 3 and hence has only a finite number of rational points. We found the rational points $(t, T) = (1, \pm 5632)$ for the equation $T^2 = q(t)$ which were overlooked in [Mal88]. Thus the Galois group G of the (separable) polynomial $f(x) := h(x, 1) \in \mathbb{Q}[x]$ can be identified with a subgroup of $\text{PSL}_3(\mathbb{F}_4)$.

The polynomial $f(x)$ has degree 21 and its leading coefficient is divisible only by the primes 2, 11, 6011 and 48481. The discriminant of $f(x)$ equals $2^{1700} \cdot 3^{200} \cdot 11^{230} \cdot 23^{12} \cdot 67^{28}$. One can check that $f(x) \bmod 5$ factors into four irreducible polynomials of degree 5 and one linear term, and that $f(x) \bmod 31$ factors into three irreducible polynomials of degree 7. Therefore, G contains elements of order 5 and of order 7. However, no maximal subgroup of $\text{PSL}_3(\mathbb{F}_4)$ has cardinality divisible by $5 \cdot 7 = 35$. Therefore, $G \cong \text{PSL}_3(\mathbb{F}_4)$.

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