EXPLICIT CLASS FIELD THEORY FOR GLOBAL FUNCTION FIELDS

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ABSTRACT. Let F be a global function field and let F^{ab} be its maximal abelian extension. Following an approach of D. Hayes, we shall construct a continuous homomorphism $\rho\colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to C_F$, where C_F is the idele class group of F. Using class field theory, we shall show that our ρ is an isomorphism of topological groups whose inverse is the Artin map of F. As a consequence of the construction of ρ, we obtain an explicit description of F^{ab}. Fix a place ∞ of F, and let A be the subring of F consisting of those elements which are regular away from ∞. We construct ρ by combining the Galois action on the torsion points of a suitable Drinfeld A-module with an associated ∞-adic representation studied by J.-K. Yu.

In the memory of David Hayes

1. INTRODUCTION

Let F be a global field, that is, a finite field extension of either $\mathbb{Q}$ or a function field $\mathbb{F}_p(t)$. Fix an algebraic closure $\overline{F}$ of F, and let F^{sep} be the separable closure of F in $\overline{F}$. *Class field theory* gives a description of the maximal abelian extension F^{ab} of F in F^{sep}. Let $\theta_F\colon C_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ be the Artin map where C_F is the idele class group of F; see §1.5 for notation and [Tat67] for background on class field theory. The map θ_F is a continuous homomorphism. By taking profinite completions, we obtain an isomorphism of topological groups:

$$\widehat{\theta}_F\colon \widehat{C}_F \xrightarrow{\sim} \mathrm{Gal}(F^{\mathrm{ab}}/F).$$

So θ_F gives a one-to-one correspondence between the finite abelian extensions L of F in F^{sep} and the finite index open subgroups of C_F. For a finite abelian extension L/F, the corresponding open subgroup of C_F is the kernel U of the homomorphism

$$C_F \xrightarrow{\theta_F} \mathrm{Gal}(F^{\mathrm{ab}}/F) \twoheadrightarrow \mathrm{Gal}(L/F)$$

where the second homomorphism is the restriction map $\sigma \mapsto \sigma|_L$ (the group U is computable; it equals $N_{L/F}(C_L)$ where $N_{L/F}\colon C_L \to C_F$ is the norm map). However, given an open subgroup U with finite index in C_F, the Artin map θ_F does *not* explicitly produce the corresponding extension field L/F (in the sense, that it does not give a concrete set of generators for L over F; though it does give enough information to uniquely characterize it). *Explicit class field theory* for F entails giving constructions for all the abelian extensions of F. We shall give a construction of F^{ab}, for a global function field F, that at the same time produces the inverse of $\widehat{\theta}_F$ (without referring to $\widehat{\theta}_F$ or class field theory itself).

1.1. Context. David Hayes [Hay74] provided an explicit class field theory for rational function fields $F = k(t)$. He built on and popularized the work of Carlitz from the 1930s [Car38], who had constructed a large class of abelian extensions using what is now called the *Carlitz module* (this is the prototypical Drinfeld module; we will recall the basic definitions concerning Drinfeld modules in §2). Drinfeld and Hayes have both described explicit class field theory for an arbitrary global function field F, see [Dri74] and [Hay79]. Both proceed by first choosing a distinguished place ∞

of F, and their constructions give the maximal abelian extension K_∞ of F that splits completely at ∞. Drinfeld defines a moduli space of rank 1 Drinfeld (elliptic) modules with level structure arising from the "finite" places of F whose spectrum turns out to be K_∞. Hayes fixes a normalized Drinfeld module ϕ of rank 1 whose field of definition along with its torsion points can be used to construct K_∞ (this approach is more favourable for explicit computations).

One approach to computing the full field F^{ab} is to chose a second place ∞' of F, since F^{ab} will then be the compositum of K_∞ and $K_{\infty'}$. We wish to give a version of explicit class field theory that does not require this second unnatural choice. In Drinfeld's second paper on Drinfeld modules [Dri77], he achieves exactly this by considering another moduli space of rank 1 Drinfeld modules but now with additional ∞-adic structure. As remarked by Goss in [Gos96, §7.5], it would be very useful to have a modification of Drinfeld's construction that can be applied directly to ϕ to give the full abelian closure F^{ab}. We shall do exactly this! J.-K. Yu [Yu03] has studied the additional ∞-adic structure introduced by Drinfeld and has teased out the implicit Galois representation occurring there. Yu's representation, which may also be defined for higher rank Drinfeld modules, can be viewed as an analogue of the Sato-Tate law, cf. [Yu03] and [Zyw11].

1.2. **Overview.**
The goal of this paper is to given an explicit construction of the inverse $\widehat{\theta}_F$. Moreover, we will construct an isomorphism of topological groups

$$\rho\colon W_F^{\mathrm{ab}} \to C_F,$$

where W_F^{ab} is the subgroup of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ that acts on the algebraic closure $\bar{k}$ of k in F^{sep} as an integral power of Frobenius map $x \mapsto x^q$ (we endow W_F^{ab} with the weakest topology for which $\mathrm{Gal}(F^{\mathrm{ab}}/\bar{k})$ is an open subgroup and has its usual profinite topology). The inverse of our ρ will be the homomorphism $\alpha \mapsto \theta_F(\alpha)^{-1}$ (Theorem 3.5). For an open subgroup U of C_F with finite index, define the homomorphism

$$\rho_U\colon W_F^{\mathrm{ab}} \to C_F \twoheadrightarrow C_F/U;$$

it factors through $\mathrm{Gal}(L_U/F)$ where L_U is the field corresponding to U via class field theory. Everything about ρ is computable, and in particular one can find generators for L_U.

In §2, we shall give the required background on Drinfeld modules; in particular, we focus on normalized Drinfeld modules of rank 1. The representation ρ will then be defined in §3.3.

The rest of the introduction serves as further motivation and will not be needed later. After a quick recap of explicit class field for $\mathbb{Q}$, we shall describe the abelian extensions of $F = k(t)$ constructed by Carlitz which will lead to a characterization of ρ_∞. We will treat the case $F = k(t)$ in more depth in §5 and recover Hayes' description of $k(t)^{\mathrm{ab}}$ as the compositum of three linearly disjoint fields.

1.3. **The rational numbers.**
We briefly recall explicit class field theory for $\mathbb{Q}$. For each positive integer n, let μ_n be the subgroup of n-torsion points in $\overline{\mathbb{Q}}^\times$; it is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 1. Let

$$\chi_n\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^\times$$

be the representation for which $\sigma(\zeta) = \zeta^{\chi_n(\sigma)}$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ and $\zeta \in \mu_n$. By taking the inverse image over all n, ordered by divisibility, we obtain a continuous and surjective representation

$$\chi\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^\times.$$

The *Kronecker-Weber theorem* says that $\mathbb{Q}^{\mathrm{ab}}$ is the cyclotomic extension of $\mathbb{Q}$, and hence χ is an isomorphism of topological groups.

The group $\widehat{\mathbb{Z}}^\times \times \mathbb{R}^+$ can be viewed as a subgroup of $\mathbb{A}_{\mathbb{Q}}^\times$, where $\mathbb{R}^+$ is the group of positive elements of $\mathbb{R}^\times$. The quotient map $\widehat{\mathbb{Z}}^\times \times \mathbb{R}^+ \to C_{\mathbb{Q}}$ is an isomorphism of topological groups, and

by taking profinite completions we obtain an isomorphism $\widehat{\mathbb{Z}}^\times = \widehat{\mathbb{Z}}^\times \times \widehat{\mathbb{R}^+} \xrightarrow{\sim} \widehat{C}_\mathbb{Q}$. Composing χ with this map, we have a isomorphism

$$\rho\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \xrightarrow{\sim} \widehat{C}_\mathbb{Q}.$$

One can show that the inverse of the Artin map $\widehat{\theta}_\mathbb{Q}\colon \widehat{C}_\mathbb{Q} \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ is simply the map $\sigma \mapsto \rho(\sigma)^{-1}$.

1.4. **The rational function field.** Let us briefly consider the rational function field $F = k(t)$ where k is a finite field with q elements; it is the function field of the projective line $\mathbb{P}^1_k$. Let ∞ denote the place of F that has uniformizer t^{-1}. The subring of F consisting of functions that are regular at all places except possibly ∞ is $A = k[t]$.

Let $\mathrm{End}_k(\mathbb{G}_{a,F})$ be the ring of k-linear endomorphisms of the additive group scheme $\mathbb{G}_a$ over F. More concretely, $\mathrm{End}_k(\mathbb{G}_{a,F})$ is the ring of polynomials $\sum_i c_i X^{q^i} \in F[X]$ with the usual addition and the multiplication operation being composition. The Carlitz module is the homomorphism

$$\phi\colon A \to \mathrm{End}_k(\mathbb{G}_{a,F}),\ a \mapsto \phi_a$$

of k-algebras for which $\phi_t = tX + X^q$. Using the Carlitz module, we can give F^{sep} an interesting new A-module structure; i.e., for $a \in A$ and $\xi \in F^{\mathrm{sep}}$, we define $a \cdot \xi := \phi_a(\xi)$.

For a monic polynomial $m \in A$, let $\phi[m]$ be the m-torsion subgroup of F^{sep}, i.e., the set of $\xi \in F^{\mathrm{sep}}$ for which $m \cdot \xi = 0$ (equivalently, the roots of the separable polynomial $\phi_m \in F[X]$). The group $\phi[m]$ is free of rank 1 as an $A/(m)$-module, and we have a continuous surjective homomorphism

$$\chi_m\colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to (A/(m))^\times$$

such that $\sigma(\xi) = \chi_m(\sigma) \cdot \xi$ for all $\sigma \in \mathrm{Gal}(F^{\mathrm{ab}}/F)$ and $\xi \in \phi[m]$. By taking the inverse image over all monic $m \in A$, ordered by divisibility, we obtain a surjective continuous representation

$$\chi\colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \widehat{A}^\times.$$

However, unlike the cyclotomic case, the map χ is not an isomorphism. The field $\bigcup_m F(\phi[m])$ is a geometric extension of F that is tamely ramified at ∞; in particular, it does not contain extensions of F that are wildly ramified at ∞ or the constant extensions.

Following J.K. Yu, and Drinfeld, we shall define a continuous homomorphism

$$\rho_\infty\colon W_F^{\mathrm{ab}} \to F_\infty^\times$$

where $F_\infty = k((t^{-1}))$ is the completion of F at the place ∞. We will put off its definition, and simply note that ρ_∞ can be characterized by the fact that it satisfies $\rho_\infty(\mathrm{Frob}_\mathfrak{p}) = \mathfrak{p}$ for each monic irreducible polynomial $\mathfrak{p}$ of A. The image of ρ_∞ is thus contained in the open subgroup $F_\infty^+ := \langle t \rangle \cdot (1 + t^{-1}k[\![t^{-1}]\!])$ of $F_\infty^\times$. We can view $\widehat{A}^\times \times F_\infty^+$ as an open subgroup of the ideles $\mathbb{A}_F^\times$. We define the continuous homomorphism

$$\rho\colon W_F^{\mathrm{ab}} \xrightarrow{\chi \times \rho_\infty} \widehat{A}^\times \times F_\infty^+ \to C_F$$

where the first map takes $\sigma \in W_F^{\mathrm{ab}}$ to $(\chi(\sigma), \rho_\infty(\sigma))$ and the second map is the compositum of the inclusion map to $\mathbb{A}_F^\times$ with the quotient map $\mathbb{A}_F^\times \to C_F$.

The main result of this paper, for $F = k(t)$, says that the above homomorphism $\rho\colon W_F^{\mathrm{ab}} \to C_F$ is an isomorphism of topological groups. Moreover, the inverse of the Artin map $\theta_F\colon C_F \to W_F^{\mathrm{ab}}$ is the homomorphism $\sigma \mapsto \rho(\sigma)^{-1}$. In particular, observe that ρ does not depend on our initial choice of place ∞! Taking profinite completions, we obtain an isomorphism $\widehat{\rho}\colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \xrightarrow{\sim} \widehat{C}_F$.

After first developing the theory for a general global function field, we will return to the case $F = k(t)$ in §5 where we will explicitly describe the abelian extension of $k(t)$ arising from ρ_∞.

The constructions for a general global function field F are more involved; they more closely resemble the theory of complex multiplication for elliptic curves than the cyclotomic theory. We first choose a place ∞ of F. In place of the Carlitz module, we will consider a suitable rank 1 Drinfeld module ϕ. We cannot always take ϕ to be defined over F, but we can choose a model defined over the maximal abelian extension H_A of F that is unramified at all places and splits completely at ∞ (we will actually work with a slightly larger field H_A^+).

1.5. **Notation.** Throughout this paper, we consider a global function field F with a fixed place ∞. Let A be the subring consisting of those functions that are regular away from ∞. Denote by k the field of constants of F and let q be the cardinality of k.

For each place λ of F, let F_λ be the completion of F at λ. Let $\mathrm{ord}_\lambda \colon F_\lambda^\times \twoheadrightarrow \mathbb{Z}$ be the discrete valuation corresponding to λ and let $\mathcal{O}_\lambda \subseteq F_\lambda$ be its valuation ring. The idele group $\mathbb{A}_F^\times$ of F is the subgroup of $(\alpha_\lambda) \in \prod_\lambda F_\lambda^\times$, where the product is over all places of F, such that α_λ belongs to $\mathcal{O}_\lambda^\times$ for all but finitely many places λ. The group $\mathbb{A}_F^\times$ is locally compact when endowed with the weakest topology for which $\prod_\lambda \mathcal{O}_\lambda^\times$, with the profinite topology, is an open subgroup. We embed $F^\times$ diagonally into $\mathbb{A}_F^\times$; it is a discrete subgroup. The idele class group of F is $C_F := \mathbb{A}_F^\times / F^\times$ which we endow with the quotient topology.

Let $\mathfrak{m}_\lambda$ be the maximal ideal of $\mathcal{O}_\lambda$. Define the finite field $\mathbb{F}_\lambda = \mathcal{O}_\lambda / \mathfrak{m}_\lambda$ whose cardinality we will denote by $N(\lambda)$. The degree of the place ∞ is $d_\infty := [\mathbb{F}_\infty : k]$.

Take a place $\mathfrak{p}$ of F. We denote by $\mathrm{Frob}_\mathfrak{p}$ an arbitrary element of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ such that for each finite Galois extension $K \subseteq F^{\mathrm{sep}}$ of F for which $\mathfrak{p}$ is unramified, $\mathrm{Frob}_\mathfrak{p}|_K$ is contained in the (arithmetic) Frobenius conjugacy class of $\mathfrak{p}$ in $\mathrm{Gal}(K/F)$.

Let L be an extension field of k. We fix an algebraic closure $\overline{L}$ of L and let L^{sep} be the separable closure of L in $\overline{L}$. We shall take $\bar{k}$ to be the algebraic closure of k in L^{sep}. Let L^{perf} be the perfect closure of L in $\overline{L}$. We shall denote the absolute Galois group of L by $\mathrm{Gal}_L := \mathrm{Gal}(L^{\mathrm{sep}}/L)$. The Weil group W_L of L is the subgroup of Gal_L consisting of those automorphisms σ for which there exists an integer $\deg(\sigma)$ that satisfy $\sigma(x) = x^{q^{\deg(\sigma)}}$ for all $x \in \bar{k}$. The map $\deg \colon W_L \to \mathbb{Z}$ is a group homomorphism with kernel $\mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k})$. We endow W_L with the weakest topology for which $\mathrm{Gal}(L^{\mathrm{sep}}/L\bar{k})$ is an open subgroup with its usual profinite topology. Let W_L^{ab} be the image of W_L under the restriction map $\mathrm{Gal}_L \to \mathrm{Gal}(L^{\mathrm{ab}}/L)$ where L^{ab} is the maximal abelian extension of L in L^{sep}.

Let L be an extension field of k. We define $L[\tau]$ be the ring of polynomials in τ with coefficients in L that obey the commutation rule $\tau \cdot a = a^q \tau$ for $a \in L$. In particular, note that $L[\tau]$ will be non-commutative if $L \neq k$. We can identify $L[\tau]$ with the k-algebra $\mathrm{End}_k(\mathbb{G}_{a,L})$ consisting of the k-linear endomorphism of the additive group scheme $\mathbb{G}_{a,L}$; identify τ with the endomorphism $X \mapsto X^q$.

Suppose that L is perfect. Let $L((\tau^{-1}))$ be the skew-field consisting of Laurent series in τ^{-1}; it contains $L[\tau]$ as a subring (we need L to be perfect so that $\tau^{-1} \cdot a = a^{1/q}\tau$ is always valid). Define the valuation $\mathrm{ord}_{\tau^{-1}} \colon L((\tau^{-1})) \to \mathbb{Z} \cup \{+\infty\}$ by $\mathrm{ord}_{\tau^{-1}}(\sum_i a_i \tau^{-i}) = \inf\{i : a_i \neq 0\}$ and $\mathrm{ord}_{\tau^{-1}}(0) = +\infty$. The valuation ring of $\mathrm{ord}_{\tau^{-1}}$ is $L[\![\tau^{-1}]\!]$, i.e., the ring of formal power series in τ^{-1}. Again note that $L((\tau^{-1}))$ and $L[\![\tau^{-1}]\!]$ are non-commutative if $L \neq k$.

For a topological group G, we will denote by $\widehat{G}$ the profinite completion of G. We will always consider profinite groups, for example Galois groups, with their profinite topology.

2. BACKGROUND

For an in-depth introduction to Drinfeld modules, see [Dri74, DH87, Gos96]. The introduction [Hay92] of Hayes and Chapter VII of [Gos96] are particularly relevant to the material of this section.

2.1. Drinfeld modules.

Let L be a field extension of k. A Drinfeld module over L is a homomorphism $\phi \colon A \to L[\tau]$, $x \mapsto \phi_x$ of k-algebras whose image contains a non-constant polynomial. Let $\partial \colon L[\tau] \to L$ be the ring homomorphism that takes a twisted polynomial to its constant term. We say that ϕ has generic characteristic if the homomorphism $\partial \circ \phi \colon A \to L$ is injective; using this map, we then obtain an embedding $F \hookrightarrow L$ that we will view as an inclusion.

The ring $L[\tau]$ is contained in the skew field $L^{\mathrm{perf}}((\tau^{-1}))$. The map ϕ is injective, so it extends uniquely to a homomorphism $\phi \colon F \hookrightarrow L^{\mathrm{perf}}((\tau^{-1}))$. The function $v \colon F \to \mathbb{Z} \cup \{+\infty\}$ defined by $v(x) = \mathrm{ord}_{\tau^{-1}}(\phi_x)$ is a discrete valuation that satisfies $v(x) \leq 0$ for all non-zero $x \in A$; the valuation v is non-trivial since the image of ϕ contain a non-constant element of $L[\tau]$. Therefore, v is equivalent to ord_∞ and hence there exists a positive $r \in \mathbb{Q}$ that satisfies

$$(2.1) \qquad \mathrm{ord}_{\tau^{-1}}(\phi_x) = r d_\infty \, \mathrm{ord}_\infty(x)$$

for all $x \in F^\times$. The number r is called the rank of ϕ and it is always an integer.

Since $L^{\mathrm{perf}}((\tau^{-1}))$ is complete with respect to $\mathrm{ord}_{\tau^{-1}}$, the map ϕ extends uniquely to a homomorphism

$$\phi \colon F_\infty \hookrightarrow L^{\mathrm{perf}}((\tau^{-1}))$$

that satisfies (2.1) for all $x \in F_\infty^\times$. This extension of ϕ was first constructed in [Dri77] and will be the key to the ∞-adic part of the construction of the inverse of the Artin map in §3.2. It will also lead to a more straightforward definition of the "leading coefficient" map μ_ϕ of [Hay92, §6], see §2.2.

Restricting our extended map ϕ to $\mathbb{F}_\infty$ gives a homomorphism $\mathbb{F}_\infty \to L^{\mathrm{perf}}[\![\tau^{-1}]\!]$. After composing $\phi|_{\mathbb{F}_\infty}$ with the homomorphism $L^{\mathrm{perf}}[\![\tau^{-1}]\!] \to L^{\mathrm{perf}}$ which takes a power series in τ^{-1} to its constant term, we obtain a homomorphism $\mathbb{F}_\infty \hookrightarrow L^{\mathrm{perf}}$ of k-algebras whose image must lie in L. In particular, the Drinfeld module ϕ gives L the structure of an $\mathbb{F}_\infty$-algebra.

Fix two Drinfeld modules $\phi, \phi' \colon A \to L[\tau]$. An isogeny from ϕ to ϕ' over L is a non-zero $f \in L[\tau]$ for which $f\phi_a = \phi'_a f$ for all $a \in A$. An isomorphism from ϕ to ϕ' over L is an $f \in L[\tau]^\times = L^\times$ that is an isogeny from ϕ to ϕ'.

2.2. Normalized Drinfeld modules.

Fix a Drinfeld module $\phi \colon A \to L[\tau]$ of rank r and also denote by ϕ its extension $F_\infty \to L^{\mathrm{perf}}((\tau^{-1}))$. For each $x \in F_\infty^\times$, we define $\mu_\phi(x) \in (L^{\mathrm{perf}})^\times$ to be the first non-zero coefficient of the Laurent series $\phi_x \in L^{\mathrm{perf}}((\tau^{-1}))$. By (2.1), the first term of ϕ_x is $\mu_\phi(x)\tau^{-r d_\infty \mathrm{ord}_\infty(x)}$. For a non-zero $x \in A$, one can also define $\mu_\phi(x)$ as the leading coefficient of ϕ_x as a polynomial in τ.

For $x, y \in F_\infty^\times$, the value $\mu_\phi(xy)$ is equal to the coefficient of $\mu_\phi(x)\tau^{-r d_\infty \mathrm{ord}_\infty(x)} \cdot \mu_\phi(y)\tau^{-r d_\infty \mathrm{ord}_\infty(y)}$, and hence

$$(2.2) \qquad \mu_\phi(xy) = \mu_\phi(x)\mu_\phi(y)^{1/q^{r d_\infty \mathrm{ord}_\infty(x)}}.$$

With respect to our embedding $\mathbb{F}_\infty \hookrightarrow L$ arising from ϕ, we have $\mu_\phi(x) = x$ for all $x \in \mathbb{F}_\infty^\times$.

We say that ϕ is normalized if $\mu_\phi(x)$ belongs to $\mathbb{F}_\infty^\times$ for all $x \in F_\infty^\times$ (equivalently, for all non-zero $x \in A$). If ϕ is normalized, then by (2.2) the map $\mu_\phi \colon F_\infty^\times \to \mathbb{F}_\infty^\times$ is a group homomorphism that equals the identity map when restricted to $\mathbb{F}_\infty^\times$; this is an example of a sign function.

Definition 2.1. A sign function for F_∞ is a group homomorphism $\varepsilon \colon F_\infty^\times \to \mathbb{F}_\infty^\times$ that is the identity map when restricted to $\mathbb{F}_\infty^\times$. We say that ϕ is ε-normalized if it is normalized and $\mu_\phi \colon F_\infty^\times \to \mathbb{F}_\infty^\times$ is equal to ε composed with some k-automorphism of F_∞.

A sign function ε is trivial on $1 + \mathfrak{m}_\infty$, so it determined by the value $\varepsilon(\pi)$ for a fixed uniformizer π of F_∞.

Lemma 2.2. [Hay92, §12] *Let ε be a sign function for F_∞ and let $\phi' \colon A \to L[\tau]$ be a Drinfeld module. Then ϕ' is isomorphic over $\bar{L}$ to an ε-normalized Drinfeld module $\phi \colon A \to \bar{L}[\tau]$.*

2.3. **The action of an ideal on a Drinfeld module.** Fix a Drinfeld module $\phi \colon A \to L[\tau]$ and take a non-zero ideal $\mathfrak{a}$ of A. Let $I_{\mathfrak{a},\phi}$ be the *left* ideal in $L[\tau]$ generated by the set $\{\phi_a : a \in \mathfrak{a}\}$. All left ideals of $L[\tau]$ are principal, so $I_{\mathfrak{a},\phi} = L[\tau] \cdot \phi_\mathfrak{a}$ for a unique monic polynomial $\phi_\mathfrak{a} \in L[\tau]$. Using that $\mathfrak{a}$ is an ideal, we find that $I_{\mathfrak{a},\phi}\phi_x \subseteq I_{\mathfrak{a},\phi}$ for all $x \in A$. Thus for each $x \in A$, there is a unique polynomial $(\mathfrak{a} * \phi)_x$ in $L[\tau]$ that satisfies

$$(2.3) \qquad\qquad \phi_\mathfrak{a} \cdot \phi_x = (\mathfrak{a} * \phi)_x \cdot \phi_\mathfrak{a}.$$

The map

$$\mathfrak{a} * \phi \colon A \to L[\tau], \ x \mapsto (\mathfrak{a} * \phi)_x$$

is also a Drinfeld module, and hence (2.3) shows that $\phi_\mathfrak{a}$ is an isogeny from ϕ to $\mathfrak{a} * \phi$.

Lemma 2.3. [Hay92, §4]
   (i) *Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero ideals of A. Then $\phi_{\mathfrak{a}\mathfrak{b}} = (\mathfrak{b} * \phi)_\mathfrak{a} \cdot \phi_\mathfrak{b}$ and $\mathfrak{a} * (\mathfrak{b} * \phi) = (\mathfrak{a}\mathfrak{b}) * \phi$.*
   (ii) *Let $\mathfrak{a} = wA$ be a non-zero principal ideal of A. Then $\phi_\mathfrak{a} = \mu_\phi(w)^{-1} \cdot \phi_w$ and $(\mathfrak{a} * \phi)_x = \mu_\phi(w)^{-1} \cdot \phi_x \cdot \mu_\phi(w)$ for all $x \in A$.*

Lemma 2.4. *Let $\sigma \colon L \hookrightarrow L'$ be an embedding of fields. Let $\sigma(\phi) \colon A \to L'[\tau]$ be the Drinfeld module for which $\sigma(\phi)_x = \sigma(\phi_x)$, where σ acts on the coefficients of $L[\tau]$. For each non-zero ideal $\mathfrak{a}$ of A, we have $\sigma(\mathfrak{a} * \phi) = \mathfrak{a} * \sigma(\phi)$ and $\sigma(\phi_\mathfrak{a}) = \sigma(\phi)_\mathfrak{a}$.*

Proof. The left ideal of $L'[\tau]$ generated by $\sigma(I_{\mathfrak{a},\phi})$ is $I_{\mathfrak{a},\sigma(\phi)}$, and hence $\sigma(\phi_\mathfrak{a}) = \sigma(\phi)_\mathfrak{a}$. Applying σ to (2.3), we have

$$\sigma(\phi)_\mathfrak{a}\sigma(\phi)_x = \sigma(\phi_\mathfrak{a} \cdot \phi_x) = \sigma((\mathfrak{a} * \phi)_x \cdot \phi_\mathfrak{a}) = \sigma(\mathfrak{a} * \phi)_x \sigma(\phi)_\mathfrak{a},$$

for all $x \in A$. This shows that $\sigma(\mathfrak{a} * \phi) = \mathfrak{a} * \sigma(\phi)$. $\qquad\qquad\square$

2.4. **Hayes modules.** Fix a sign function ε for F_∞. Let $\mathbb{C}_\infty$ be the completion of an algebraic closure $\bar{F}_\infty$ of F_∞ with respect to the ∞-adic norm; it is both complete and algebraically closed.

Definition 2.5. A Hayes module for ε is a Drinfeld module $\phi \colon A \to \mathbb{C}_\infty[\tau]$ of rank 1 that is ε-normalized and for which $\partial \circ \phi \colon A \to \mathbb{C}_\infty$ is the inclusion map. Denote by X_ε the set of Hayes modules for ε.

We know that Hayes modules exist because Drinfeld A-modules over $\mathbb{C}_\infty$ of rank 1 can be constructed analytically [Dri74, §3] and then we can apply Lemma 2.2.[1]

Take any Hayes module $\phi \in X_\varepsilon$. Using that $\phi_\mathfrak{a} \in \mathbb{C}_\infty[\tau]$ is monic along with (2.3), we see that the Drinfeld module $\mathfrak{a} * \phi$ also belongs to X_ε. By Lemma 2.3(i), we find that the group $\mathcal{I}$ of fractional ideals of A acts on X_ε. Let $\mathcal{P}^+$ be the subgroup of principal fractional ideals generated by those $x \in F^\times$ that satisfy $\varepsilon(x) = 1$. The group $\mathcal{P}^+$ acts trivially on X_ε by Lemma 2.3(ii), and hence induces an action of the finite group $\mathrm{Pic}^+(A) := \mathcal{I}/\mathcal{P}^+$ on X_ε.

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[1]In our construction of the inverse of the Artin map, this is the only part that we have not made explicit. It is analogous to how one analytically constructs an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_K$ of a quadratic imaginary field K [The quotient $\mathbb{C}/\mathcal{O}_K$ gives such an elliptic curve over $\mathbb{C}$. We can then compute the j-invariant to high enough precision to identify what algebraic integer it is (it belongs to the Hilbert class field of K and we know its degree over $\mathbb{Q}$).]. The current version of Magma [BCP97] has a function `AnalyticDrinfeldModule` that can compute rank 1 Drinfeld modules that are defined over the maximal abelian extension H_A of F that is unramified at all places and splits completely at ∞.

Proposition 2.6. [Hay92, §13] *The set X_ε is a principal homogeneous space for $\mathrm{Pic}^+(A)$ under the $*$ action.*

We now consider the arithmetic of the set X_ε. Take any $\phi \in X_\varepsilon$ and choose a non-constant $y \in A$. Let H_A^+ be the subfield of $\mathbb{C}_\infty$ generated by F and the coefficients of ϕ_y as a polynomial in τ. We call H_A^+ the **normalizing field** for the triple (F, ∞, ε).

Lemma 2.7. [Hay92, §14] *The extension H_A^+/F is finite and normal, and depends only on the triple (F, ∞, ε).*

So for every $\phi \in X_\varepsilon$, we have $\phi(A) \subseteq H_A^+[\tau]$. From now on, we shall view ϕ as a Drinfeld module $\phi \colon A \to H_A^+[\tau]$. By [Hay92, Proposition 11.4], we actually have $\phi(A) \subseteq B[\tau]$ where B is the integral closure of A in H_A^+. Since ϕ is normalized, we find that ϕ has **good reduction** at every place of H_A^+ not lying over ∞ (for each non-zero prime ideal $\mathfrak{P}$ of B, we can compose ϕ with a reduction modulo $\mathfrak{P}$ map to obtain a Drinfeld module of rank 1 over $B/\mathfrak{P}$).

There is thus a natural action of the Galois group $\mathrm{Gal}(H_A^+/F)$ on X_ε. With a fixed $\phi \in X_\varepsilon$, Proposition 2.6 implies that there is a unique function $\psi \colon \mathrm{Gal}(H_A^+/F) \hookrightarrow \mathrm{Pic}^+(A)$ such that $\sigma(\phi) = \psi(\sigma) * \phi$. Lemmas 2.3 and 2.4 imply that ψ is a group homomorphism that does not depend on the initial choice of ϕ. A consequence of following important proposition is that ψ is surjective, and hence an isomorphism.

Proposition 2.8. *The extension H_A^+/F is unramified away from ∞. For each non-zero prime ideal $\mathfrak{p}$ of A, the class $\psi(\mathrm{Frob}_\mathfrak{p})$ of $\mathrm{Pic}^+(A)$ is the one containing $\mathfrak{p}$.*

3. Construction of the inverse of the Artin map

Fix a place ∞ of F. Throughout this section, we also fix a sign function ε for F_∞ and a Hayes module $\phi \in X_\varepsilon$ (as described in the previous section). Let F_∞^+ be the open subgroup of $F_\infty^\times$ consisting of those $x \in F_\infty^\times$ for which $\varepsilon(x) = 1$. So as not to clutter the construction, all the lemmas of §3 will be proved in §4.

3.1. λ-adic representations.
Fix a place $\lambda \neq \infty$ of F; we shall also denote by λ the corresponding maximal ideal of A. Take any automorphism $\sigma \in \mathrm{Gal}_F$. Since the map ψ of §2.4 is surjective, we can choose a non-zero ideal $\mathfrak{a}$ of A for which $\sigma(\phi) = \mathfrak{a} * \phi$.

For each positive integer e, let $\phi[\lambda^e]$ be the set of $b \in \overline{F}$ that satisfy $\phi_x(b) = 0$ for all $x \in \lambda^e$; equivalently, $\phi[\lambda^e]$ is the set of $b \in \overline{F}$ such that $\phi_{\lambda^e}(b) = 0$ (recall that we can identify each element of $L[\tau]$ with a unique polynomial $\sum_{i \geq 0} c_i X^{q^i} \in L[X]$). We have $\phi[\lambda^e] \subseteq F^{\mathrm{sep}}$ since the polynomials ϕ_x are separable for all $x \in \lambda^e$. Using the A-module structure coming from ϕ, we find that $\phi[\lambda^e]$ is an A/λ^e-module of rank 1. The λ-adic **Tate module** of ϕ is defined to be

$$T_\lambda(\phi) = \mathrm{Hom}_A(F_\lambda/\mathcal{O}_\lambda, \phi[\lambda^\infty])$$

where $\phi[\lambda^\infty] = \cup_{e \geq 1} \phi[\lambda^e]$. The Tate module $T_\lambda(\phi)$ is a free $\mathcal{O}_\lambda$-module of rank 1, and hence $V_\lambda(\phi) := F_\lambda \otimes_{\mathcal{O}_\lambda} T_\lambda(\phi)$ is a one-dimensional vector space over F_λ.

For each e, the map $\phi[\lambda^e] \to \sigma(\phi)[\lambda^e]$, $\xi \mapsto \sigma(\xi)$ is an isomorphism of A/λ^e-modules. Combining over all e, we obtain an isomorphism $V_\lambda(\sigma) \colon V_\lambda(\phi) \to V_\lambda(\sigma(\phi))$ of F_λ-vector spaces.

The isogeny $\phi_\mathfrak{a}$ from ϕ to $\mathfrak{a} * \phi$ induces a homomorphism $\phi[\lambda^e] \to (\mathfrak{a} * \phi)[\lambda^e]$ of A/λ^e-module for each e. Combining together, we obtain an isomorphism $V_\lambda(\phi_\mathfrak{a}) \colon V_\lambda(\phi) \to V_\lambda(\mathfrak{a} * \phi)$ of F_λ-vector spaces.

Using our assumption $\sigma(\phi) = \mathfrak{a} * \phi$, the map $V_\lambda(\phi_\mathfrak{a})^{-1} \circ V_\lambda(\sigma)$ belongs to $\mathrm{Aut}_{F_\lambda}(V_\lambda(\phi)) = F_\lambda^\times$; we denote this element of $F_\lambda^\times$ by $\rho_\lambda^\mathfrak{a}(\sigma)$.

Lemma 3.1.

(i) *Take $\sigma, \gamma \in \mathrm{Gal}_F$ and fix ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A such that $\sigma(\phi) = \mathfrak{a} * \phi$ and $\gamma(\phi) = \mathfrak{b} * \phi$. Then $(\sigma\gamma)(\phi) = (\mathfrak{a}\mathfrak{b}) * \phi$ and $\rho_\lambda^{\mathfrak{a}\mathfrak{b}}(\sigma\gamma) = \rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\gamma)$.*

(ii) *Take $\sigma \in \mathrm{Gal}_F$ and fix ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A such that $\sigma(\phi) = \mathfrak{a} * \phi = \mathfrak{b} * \phi$. Then $\rho_\lambda^{\mathfrak{a}}(\sigma)\rho_\lambda^{\mathfrak{b}}(\sigma)^{-1}$ is the unique generator $w \in F^\times$ of the fractional ideal $\mathfrak{b}\mathfrak{a}^{-1}$ that satisfies $\varepsilon(w) = 1$.*

(iii) *Take $\sigma \in \mathrm{Gal}_F$ and fix an ideal $\mathfrak{a}$ such that $\sigma(\phi) = \mathfrak{a} * \phi$. Identifying λ with a non-zero prime ideal of A, let $f \geq 0$ be the largest power of λ dividing $\mathfrak{a}$. Then $\mathrm{ord}_\lambda(\rho_\lambda^{\mathfrak{a}}(\sigma)) = -f$.*

By Lemma 3.1(i), the map

$$\rho_\lambda \colon \mathrm{Gal}_{H_A^+} \to \mathcal{O}_\lambda^\times, \quad \sigma \mapsto \rho_\lambda^A(\sigma)$$

is a homomorphism. It is continuous and it is unramified at all places not lying over λ or ∞ by the Drinfeld module analogue of the Néron-Ogg-Shafarevich criterion, cf. [Gos96, Theorem 4.10.5] (as mentioned after Lemma 2.7, ϕ has good reduction at all places of H_A^+ not lying over ∞).

3.2. ∞-adic representation. By §2.2, our Drinfeld module $\phi \colon A \to H_A^+[\tau]$ extends uniquely to a homomorphism $\phi \colon F_\infty \hookrightarrow (H_A^+)^{\mathrm{perf}}((\tau^{-1}))$ that satisfies (2.1) with $r = 1$ for all $x \in F_\infty^\times$. Recall that we defined a homomorphism $\deg \colon W_F \twoheadrightarrow \mathbb{Z}$ for which $\sigma(x)$ equals $x^{q^{\deg(\sigma)}} = \tau^{\deg(\sigma)} x \tau^{-\deg(\sigma)}$ for all $x \in \bar{k}$. Given a series $u \in \bar{F}((\tau^{-1}))$ with coefficients in F^{sep} and an automorphism $\sigma \in \mathrm{Gal}_F$, we define $\sigma(u)$ to be the series obtained by applying σ to the coefficients of u.

Lemma 3.2.

(i) *There exists a series $u \in \bar{F}[[\tau^{-1}]]^\times$ such that $u^{-1}\phi(F_\infty)u \subseteq \bar{k}((\tau^{-1}))$. Any such u has coefficients in F^{sep}.*

(ii) *Fix any u as in (i). Take any $\sigma \in W_F$ and fix a non-zero ideal $\mathfrak{a}$ of A for which $\sigma(\phi) = \mathfrak{a} * \phi$. Then*

$$\phi_{\mathfrak{a}}^{-1}\sigma(u)\tau^{\deg(\sigma)}u^{-1} \in \bar{F}((\tau^{-1}))$$

belongs to $\phi(F_\infty^+)$ and is independent of the choice of u.

Take $\sigma \in W_F$ and fix a non-zero ideal $\mathfrak{a}$ of A such that $\sigma(\phi) = \mathfrak{a} * \phi$. Choose a series $u \in \bar{F}[[\tau^{-1}]]^\times$ as in Lemma 3.2(i). Using that $\phi \colon F_\infty \to (H_A^+)^{\mathrm{perf}}((\tau^{-1}))$ is injective and Lemma 3.2(ii), we define $\rho_\infty^{\mathfrak{a}}(\sigma)$ to be the unique element of F_∞^+ for which

$$\phi\bigl(\rho_\infty^{\mathfrak{a}}(\sigma)\bigr) = \phi_{\mathfrak{a}}^{-1}\sigma(u)\tau^{\deg(\sigma)}u^{-1}.$$

We now state some results about $\rho_\infty^{\mathfrak{a}}(\sigma)$; they are analogous to those concerning $\rho_\lambda^{\mathfrak{a}}(\sigma)$ in the previous section.

Lemma 3.3.

(i) *Take $\sigma, \gamma \in W_F$ and fix ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A such that $\sigma(\phi) = \mathfrak{a} * \phi$ and $\gamma(\phi) = \mathfrak{b} * \phi$. Then $(\sigma\gamma)(\phi) = (\mathfrak{a}\mathfrak{b}) * \phi$, and we have $\rho_\infty^{\mathfrak{a}\mathfrak{b}}(\sigma\gamma) = \rho_\infty^{\mathfrak{a}}(\sigma)\rho_\infty^{\mathfrak{b}}(\gamma)$.*

(ii) *Take $\sigma \in W_F$ and fix ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A such that $\sigma(\phi) = \mathfrak{a} * \phi = \mathfrak{b} * \phi$. Then $\rho_\infty^{\mathfrak{a}}(\sigma)\rho_\infty^{\mathfrak{b}}(\sigma)^{-1}$ is the unique generator $w \in F^\times$ of the fractional ideal $\mathfrak{b}\mathfrak{a}^{-1}$ that satisfies $\varepsilon(w) = 1$.*

Lemma 3.4. *The map*

$$\rho_\infty \colon W_{H_A^+} \to F_\infty^+, \quad \sigma \mapsto \rho_\infty^A(\sigma)$$

is a continuous homomorphism that is unramified at all places of H_A^+ not lying over ∞.

3.3. The inverse of the Artin map.

For each $\sigma \in W_F$, fix a non-zero ideal $\mathfrak{a}$ of A such that $\sigma(\phi) = \mathfrak{a} * \phi$. By Lemma 3.1(iii), $(\rho_\lambda^{\mathfrak{a}}(\sigma))_\lambda$ is an idele of F. We define $\rho(\sigma)$ to be the element of the idele class group C_F that is represented by $(\rho_\lambda^{\mathfrak{a}}(\sigma))_\lambda$. By Lemmas 3.1(ii) and 3.3(ii), we find that $\rho(\sigma)$ is independent of the choice of $\mathfrak{a}$. Lemmas 3.1(i) and 3.3(i) imply that the map

$$\rho \colon W_F \to C_F$$

is a group homomorphism. The restriction of ρ to the finite index open subgroup $W_{H_A^+}$ agrees with

$$W_{H_A^+} \xrightarrow{\Pi_\lambda \rho_\lambda} F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \hookrightarrow C_F$$

where the second homomorphism is obtained by composing the natural inclusion into $\mathbb{A}_F^\times$ with the quotient map $\mathbb{A}_F^\times \to C_F$. Since the representations ρ_λ are continuous, we deduce that $\prod_\lambda \rho_\lambda$, and hence ρ, is continuous. So we may view ρ as a continuous homomorphism $W_F^{\mathrm{ab}} \to C_F$. By taking profinite completions, we obtain a continuous homomorphism

$$\widehat{\rho} \colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \widehat{C}_F.$$

Recall that the Artin map $\theta_F \colon C_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ of class field theory gives an isomorphism

$$\widehat{\theta}_F \colon \widehat{C}_F \xrightarrow{\sim} \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

of topological groups. Our main result is then the following:

Theorem 3.5. *The map $\widehat{\rho} \colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \widehat{C}_F$ is an isomorphism of topological groups. The inverse of the isomorphism $\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \widehat{C}_F$, $\sigma \mapsto \widehat{\rho}(\sigma)^{-1}$ is the Artin map $\widehat{\theta}_F$.*

Before proving the theorem, we mention the following arithmetic input.

Lemma 3.6. *Fix a place λ of F. Let $\mathfrak{p} \neq \lambda, \infty$ be a place of F which we identify with the corresponding non-zero prime ideal of A. Then $\rho_\lambda^{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}}) = 1$.*

Proof of Theorem 3.5. Take an open subgroup U of C_F with finite index. Let L_U be the fixed field in F^{sep} of the kernel of the homomorphism $W_F^{\mathrm{ab}} \xrightarrow{\rho} C_F \twoheadrightarrow C_F/U$; this gives an injective group homomorphism $\rho_U \colon \mathrm{Gal}(L_U/F) \hookrightarrow C_F/U$. Let S_U be the set of places $\mathfrak{p}$ of F for which $\mathfrak{p} = \infty$ or for which there exists an idele $\alpha \in \mathbb{A}_F^\times$ whose class in C_F does not lie in U and satisfying $\alpha_\lambda = 1$ for $\lambda \neq \mathfrak{p}$ and $\alpha_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^\times$. The set S_U is finite since U is open in C_F.

Take any place $\mathfrak{p} \notin S_U$. Choose a uniformizer $\pi_{\mathfrak{p}}$ of $F_{\mathfrak{p}}$ and let $\alpha(\mathfrak{p})$ be the idele of F that is $\pi_{\mathfrak{p}}$ at the place $\mathfrak{p}$ and 1 at all other places. Define the idele $\beta := (\rho_\lambda^{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}}))_\lambda \cdot \alpha(\mathfrak{p}) \in \mathbb{A}_F^\times$. Lemma 3.6 says that $\beta_\lambda = 1$ for all $\lambda \neq \mathfrak{p}$ while Lemma 3.1(iii) tells us that $\mathrm{ord}_{\mathfrak{p}}(\beta_{\mathfrak{p}}) = 0$. By our choice of S_U, the image of β in C_F must lie in U. Therefore, $\rho_U(\mathrm{Frob}_{\mathfrak{p}})$ is the coset of C_F/U represented by $\alpha(\mathfrak{p})^{-1}$. In particular, note that L_U/F is unramified at all $\mathfrak{p} \notin S_U$. The group C_F/U is generated by the elements $\alpha(\mathfrak{p})$ with $\mathfrak{p} \notin S_U$, and hence ρ_U is surjective. Therefore $\rho_U \colon \mathrm{Gal}(L_U/F) \to C_F/U$ is an isomorphism of groups.

Define the isomorphism $\theta_U \colon C_F/U \xrightarrow{\sim} \mathrm{Gal}(L_U/F)$, $\alpha \mapsto (\rho_U^{-1}(\alpha))^{-1}$. For each $\mathfrak{p} \notin S_U$, it takes the coset containing $\alpha(\mathfrak{p})$ to the Frobenius automorphism corresponding to $\mathfrak{p}$. Composing the quotient map $C_F \to C_F/U$ with θ_U, we find that the resulting homomorphism $C_F \to \mathrm{Gal}(L_U/F)$ equals the map $\alpha \mapsto \theta_F(\alpha)|_{L_U}$ where $\theta_F \colon C_F \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ is the Artin map of F.

Recall that class field theory gives a one-to-one correspondence between the finite abelian extensions L of F and the open subgroups U with finite index in C_F. Let $L \subseteq F^{\mathrm{sep}}$ be an arbitrary finite abelian extension of F. Class field theory says that L corresponds to the kernel U of the map $C_F \to \mathrm{Gal}(L/F)$, $\alpha \mapsto \theta_F(\alpha)|_L$. By comparing with the computation above, we deduce that $L = L_U$. Since L was an arbitrary finite abelian extension, we deduce that $F^{\mathrm{ab}} = \bigcup_U L_U$.

Taking the inverse limit of the isomorphisms $\rho_U\colon \mathrm{Gal}(L_U/F) \xrightarrow{\sim} C_F/U$ as U varies, we find that the corresponding homomorphism $\rho\colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \widehat{C}_F$ is an isomorphism (the injectivity is precisely the statement that $F^{\mathrm{ab}} = \bigcup_U L_U$). The inverse of the isomorphism $\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \widehat{C}_F$, $\sigma \mapsto \widehat{\rho}(\sigma)^{-1}$ is obtained by combining the homomorphisms $\theta_U\colon C_F/U \xrightarrow{\sim} \mathrm{Gal}(L_U/F)$; from the calculation above, this equals $\widehat{\theta}_F$. $\qquad\square$

Corollary 3.7. *The homomorphism $\rho\colon W_F^{\mathrm{ab}} \to C_F$ is an isomorphism of topological groups. The inverse of the isomorphism $W_F^{\mathrm{ab}} \to C_F$, $\sigma \mapsto \rho(\sigma)^{-1}$ is the Artin map θ_F.*

Proof. This follows directly from the theorem. Observe that the natural maps $W_F^{\mathrm{ab}} \to \widehat{W_F^{\mathrm{ab}}} = \mathrm{Gal}(F^{\mathrm{ab}}/F)$ and $C_F \to \widehat{C}_F$ from the group to their profinite completion are both injective since F is a global function field. $\qquad\square$

Remark 3.8.

   (i) The isomorphism $\rho\colon W_F^{\mathrm{ab}} \to C_F$ depends only on F (and not on our choices of ∞, ε, and $\phi \in X_\varepsilon$).
   (ii) Our proof only requires class field theory to prove that ρ is injective, i.e., to show that we have constructed all finite abelian extensions of F.

4. Proofs of the lemmas

4.1. Proof of Lemma 3.1. (i) By Lemma 2.3(i) and 2.4, we have $(\sigma\gamma)\phi = (\mathfrak{a}\mathfrak{b}) * \phi$ and $\sigma(\phi_\mathfrak{b}) = \sigma(\phi)_\mathfrak{b}$. By Lemma 2.3(i) and our choice of $\mathfrak{a}$, we find that $\phi_{\mathfrak{a}\mathfrak{b}} = (\mathfrak{a} * \phi)_\mathfrak{b} \cdot \phi_\mathfrak{a} = \sigma(\phi)_\mathfrak{b}\phi_\mathfrak{a} = \sigma(\phi_\mathfrak{b})\phi_\mathfrak{a}$. We have $\sigma(\phi_\mathfrak{b}(\sigma^{-1}(\xi))) = \sigma(\phi)_\mathfrak{b}(\xi)$ for all $\xi \in \sigma(\phi)[\lambda^e]$, so $V_\lambda(\sigma) \circ V_\lambda(\phi_\mathfrak{b}) \circ V_\lambda(\sigma)^{-1}$ and $V_\lambda(\sigma(\phi_\mathfrak{b}))$ are the same automorphism of $V_\lambda(\sigma(\phi))$. Therefore,

$$
\begin{aligned}
V_\lambda(\phi_{\mathfrak{a}\mathfrak{b}})^{-1} \circ V_\lambda(\sigma\gamma) &= V_\lambda(\phi_\mathfrak{a})^{-1} \circ V_\lambda(\sigma(\phi_\mathfrak{b}))^{-1} \circ V_\lambda(\sigma\gamma) \\
&= V_\lambda(\phi_\mathfrak{a})^{-1} \circ (V_\lambda(\sigma) \circ V_\lambda(\phi_\mathfrak{b}) \circ V_\lambda(\sigma)^{-1})^{-1} \circ V_\lambda(\sigma\gamma) \\
&= V_\lambda(\phi_\mathfrak{a})^{-1} \circ V_\lambda(\sigma) \circ V_\lambda(\phi_\mathfrak{b})^{-1} \circ V_\lambda(\sigma)^{-1} \circ V_\lambda(\sigma\gamma) \\
&= (V_\lambda(\phi_\mathfrak{a})^{-1} \circ V_\lambda(\sigma)) \circ (V_\lambda(\phi_\mathfrak{b})^{-1} \circ V_\lambda(\gamma)).
\end{aligned}
$$

Thus $\rho_\lambda^{\mathfrak{a}\mathfrak{b}}(\sigma\gamma) = \rho_\lambda^\mathfrak{a}(\sigma)\rho_\lambda^\mathfrak{b}(\gamma)$.

(ii) Since $\mathfrak{a} * \phi = \mathfrak{b} * \phi$, Lemma 2.6 implies that the fractional ideal $\mathfrak{b}\mathfrak{a}^{-1}$ is the identity class in $\mathrm{Pic}^+(A)$. There are thus non-zero $w_1, w_2 \in A$ such that $(w_1)\mathfrak{a} = (w_2)\mathfrak{b}$ and $\varepsilon(w_1) = \varepsilon(w_2) = 1$. In particular, $w := w_1/w_2$ is the unique generator of $\mathfrak{b}\mathfrak{a}^{-1}$ satisfying $\varepsilon(w) = 1$. By Lemma 2.3(ii), we have $(w_1) * \phi = \phi$ and $\phi_{(w_1)} = \phi_{w_1}$. Therefore, by Lemma 2.3(i) we have

$$
\phi_{\mathfrak{a}(w_1)} = ((w_1) * \phi)_\mathfrak{a} \cdot \phi_{(w_1)} = \phi_\mathfrak{a}\phi_{w_1}.
$$

Similarly, $\phi_{\mathfrak{b}(w_2)} = \phi_\mathfrak{b}\phi_{w_2}$, and hence $\phi_\mathfrak{a}\phi_{w_1} = \phi_\mathfrak{b}\phi_{w_2}$. Thus

$$
\rho_\lambda^\mathfrak{a}(\sigma)\rho_\lambda^\mathfrak{b}(\sigma)^{-1} = V_\lambda(\phi_\mathfrak{a})^{-1} \circ V_\lambda(\phi_\mathfrak{b}) = V_\lambda(\phi_{w_1}) \circ V_\lambda(\phi_{w_2})^{-1}.
$$

The automorphisms $V_\lambda(\phi_{w_1})$ and $V_\lambda(\phi_{w_2})$ both belong to $\mathrm{Aut}_{F_\lambda}(V_\lambda(\phi)) = F_\lambda^\times$ and correspond to w_1 and w_2, respectively. We conclude that $\rho_\lambda^\mathfrak{a}(\sigma)\rho_\lambda^\mathfrak{b}(\sigma)^{-1} = w_1 w_2^{-1} = w$.

(iii) We view λ as both a place of F and a non-zero prime ideal of A. For each $e \geq f$, the kernel of $\phi[\lambda^e] \to \phi[\lambda^e]$, $\xi \mapsto \sigma^{-1}(\phi_\mathfrak{a}(\xi))$ has cardinality $|\phi[\lambda^f]| = N(\lambda)^f$. So $\rho_\lambda^\mathfrak{a}(\sigma)^{-1} = V_\lambda(\sigma^{-1}) \circ V_\lambda(\phi_\mathfrak{a})$, which is an element of $\mathrm{Aut}_{F_\lambda}(V_\lambda(\phi)) = F_\lambda^\times$, gives an $\mathcal{O}_\lambda$-module homomorphism $T_\lambda(\phi) \to T_\lambda(\phi)$ whose cokernel has cardinality $N(\lambda)^f$. Therefore, $\mathrm{ord}_\lambda(\rho_\lambda^\mathfrak{a}(\sigma)^{-1}) = f$. In particular, we have $\mathrm{ord}_\lambda(\rho_\lambda^\mathfrak{a}(\sigma)) = -f$.

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4.2. **Proof of Lemma 3.2.** Fix a non-constant $y \in A$ that satisfies $\varepsilon(y) = 1$ and define $h :=$ $-d_\infty \operatorname{ord}_\infty(y) \geq 1$. Since ϕ is ε-normalized and $\varepsilon(y) = 1$, we have $\phi_y = \tau^h + \sum_{j=0}^{h-1} b_j \tau^j$ for unique $b_j \in H_A^+$. We set $u = \sum_{i=0}^\infty a_i \tau^{-i}$ with $a_i \in \overline{F}$ to be determined where $a_0 \neq 0$. Expanding out the series $\phi_y u$ and $u\tau^h$, we find that $\phi_y u = u\tau^h$ holds if and only if

(4.1)
$$a_i^{q^h} - a_i = - \sum_{\substack{0 \leq j \leq h-1, \\ i+j-h \geq 0}} b_j a_{i+j-h}^{q^j}$$

holds for all $i \geq 0$. We can use the equations (4.1) to recursively solve for $a_0 \neq 0, a_1, a_2, \ldots$. The a_i belong to F^{sep} since (4.1) is a separable polynomial in a_i and the b_j belong to $H_A^+ \subseteq F^{\mathrm{sep}}$. Let k_h be the degree h extension of k in $\overline{k}$. The elements of the ring $\overline{F}((\tau^{-1}))$ that commute with τ^h are $k_h((\tau^{-1}))$. Since τ^h belongs to the commutative ring $u^{-1}\phi(F_\infty)u$, we find that $u^{-1}\phi(F_\infty)u$ is a subset of $k_h((\tau^{-1}))$. Thus $u \in \overline{F}((\tau^{-1}))^\times$ has coefficients in F^{sep} and satisfies $u^{-1}\phi(F_\infty)u \subseteq \overline{k}((\tau^{-1}))$.

Recall that ϕ induces an embedding $\mathbb{F}_\infty \hookrightarrow L$; this gives inclusions $\mathbb{F}_\infty \subseteq \overline{k} \subseteq \overline{L}$. Fix a uniformizer π of F_∞. There is a unique homomorphism $\iota \colon F_\infty \to \overline{k}((\tau^{-1}))$ that satisfies the following conditions:

- $\iota(x) = x$ for all $x \in \mathbb{F}_\infty$,
- $\iota(\pi) = \tau^{-d_\infty}$,
- $\operatorname{ord}_{\tau^{-1}}(\iota(x)) = d_\infty \operatorname{ord}_\infty(x)$ for all $x \in F_\infty^\times$.

We have $\iota(F_\infty) = \mathbb{F}_\infty((\tau^{-d_\infty}))$. Let C be the centralizer of $\iota(F_\infty)$ in $\overline{F}((\tau^{-1}))$. Using that $\mathbb{F}_\infty$ and τ^{d_∞} are in $\iota(F_\infty)$, we find that $C = \mathbb{F}_\infty((\tau^{-d_\infty})) = \iota(F_\infty)$.

Take any $v \in \overline{F}[\![\tau^{-1}]\!]^\times$ that satisfies $v^{-1}\phi(F_\infty)v \subseteq \overline{k}((\tau^{-1}))$. By [Yu03, Lemma 2.3], there exist w_1 and $w_2 \in \overline{k}[\![\tau^{-1}]\!]^\times$ such that

$$w_1^{-1}(u^{-1}\phi_x u)w_1 = \iota(x) = w_2^{-1}(v^{-1}\phi_x v)w_2$$

for all $x \in F_\infty$. So for all $x \in F_\infty$, we have $(uw_1)\iota(x)(uw_1)^{-1} = \phi_x = (vw_2)\iota(x)(vw_2)^{-1}$ and hence

$$(w_2^{-1}v^{-1}uw_1)\iota(x)(w_2^{-1}v^{-1}uw_1)^{-1} = \iota(x).$$

Thus $w_2^{-1}v^{-1}uw_1$ belongs to $C \subseteq \overline{k}((\tau^{-1}))$, and so $v = uw$ for some $w \in \overline{k}[\![\tau^{-1}]\!]^\times$.

The coefficients of v thus lie in F^{sep} since the coefficients of u lie in F^{sep} and w has coefficients in the perfect field $\overline{k} \subseteq F^{\mathrm{sep}}$. This completes the proof of (i). Since w has coefficients in $\overline{k}$, we have $\sigma(w) = \tau^{\deg(\sigma)}w\tau^{-\deg(\sigma)}$ and hence

$$\sigma(v)\tau^{\deg(\sigma)}v^{-1} = \sigma(uw)\tau^{\deg(\sigma)}(uw)^{-1}$$
$$= \sigma(u)\sigma(w)\tau^{\deg(\sigma)}w^{-1}u^{-1}$$
$$= \sigma(u)(\tau^{\deg(\sigma)}w\tau^{-\deg(\sigma)})\tau^{\deg(\sigma)}w^{-1}u^{-1}$$
$$= \sigma(u)\tau^{\deg(\sigma)}u^{-1}.$$

This proves that $\phi_\mathfrak{a}^{-1}\sigma(u)\tau^{\deg(\sigma)}u^{-1}$ is independent of the initial choice of u.

We now show that $\phi_\mathfrak{a}^{-1}\sigma(u)\tau^{\deg(\sigma)}u^{-1}$ belongs to $\phi(F_\infty)$. We have seen that $\phi(F_\infty)$ is conjugate to $\iota(F_\infty)$ in $\overline{F}((\tau^{-1}))$ and that $\iota(F_\infty)$ is its own centralizer in $\overline{F}((\tau^{-1}))$. Therefore, the centralizer of $\phi(F_\infty)$ in $\overline{F}((\tau^{-1}))$ is $\phi(F_\infty)$. So it suffices to prove that $\phi_\mathfrak{a}^{-1}\sigma(u)\tau^{\deg(\sigma)}u^{-1}$ commutes with ϕ_x for all $x \in A$. Take any $x \in A$. We have $\sigma(u^{-1}\phi_x u) = \tau^{\deg \sigma}(u^{-1}\phi_x u)\tau^{-\deg \sigma}$ since $u^{-1}\phi_x u$ has coefficients in $\overline{k}$. By our choice of $\mathfrak{a}$, we have $\sigma(\phi)_x = (\mathfrak{a} * \phi)_x = \phi_\mathfrak{a}\phi_x\phi_\mathfrak{a}^{-1}$. Therefore,

$$\tau^{\deg \sigma}(u^{-1}\phi_x u)\tau^{-\deg \sigma} = \sigma(u^{-1}\phi_x u) = \sigma(u)^{-1}\sigma(\phi)_x\sigma(u) = \sigma(u)^{-1}\phi_\mathfrak{a}\phi_x\phi_\mathfrak{a}^{-1}\sigma(u)$$

and one concludes that $\phi_\mathfrak{a}^{-1}\sigma(u)\tau^{\deg \sigma}u^{-1}$ commutes with ϕ_x.

The only thing that remains to be proved is that $\phi_{\mathfrak{a}}^{-1}\sigma(u)\tau^{\deg\sigma}u^{-1}$ belongs to $\phi(F_\infty^+)$. Since ϕ is ε-normalized, this is equivalent to showing that the first non-zero coefficient of the Laurent series $\phi_{\mathfrak{a}}^{-1}\sigma(u)\tau^{\deg\sigma}u^{-1}$ in τ^{-1} is 1. Since $\phi_{\mathfrak{a}}$ is a monic polynomial of τ, we need only show that the first non-zero coefficient of $\sigma(u)\tau^{\deg\sigma}u^{-1} \in \overline{F}((\tau^{-1}))$ is 1, i.e., $\sigma(a_0)a_0^{-q^{\deg\sigma}} = 1$. This is true since $a_0 \in \bar{k}^\times$; indeed, a_0 is non-zero and satisfies $a_0^{q^h} - a_0 = 0$ by (4.1).

Remark 4.1. Take any place $\mathfrak{p} \neq \infty$ of F and any valuation $v\colon F^{\mathrm{sep}} \to \mathbb{Q}\cup\{+\infty\}$ extending $\mathrm{ord}_{\mathfrak{p}}$. We have $v(b_j) \geq 0$ for $0 \leq j \leq h$ (in §2.4, we noted that the coefficients of ϕ_y are integral over A). Using (4.1) repeatedly, we find that $v(a_i) \geq 0$ for all $i \geq 0$ (the roots of (4.1), as a polynomial in a_i, differ by a value in k_h).

For each $i \geq 1$, the extension $F(a_i)/F(a_{i-1})$ is an Artin-Schreier extension. Since the right-hand side of (4.1) is integral at each place not lying over ∞, we deduce that $F(a_i)/F(a_{i-1})$ is unramified at all places not lying over ∞.

Let $L \subseteq F^{\mathrm{sep}}$ be the extension of F generated by $\bar{k}$ and the set $\{a_i\}_{i\geq 0}$, i.e., the extension of $F\bar{k}$ generated by the coefficients of u. We find that L is unramified at all places of F away from ∞.

4.3. Proof of Lemma 3.3.
Fix a series $u \in \overline{F}[\![\tau^{-1}]\!]^\times$ as in Lemma 3.2(i).

(i) We have $\phi(\rho_\infty^{\mathfrak{ab}}(\sigma\gamma)) = \phi_{\mathfrak{ab}}^{-1}\sigma\gamma(u)\tau^{\deg(\sigma\gamma)}u^{-1}$ and $\phi(\rho_\infty^{\mathfrak{b}}(\gamma)) = \phi_{\mathfrak{b}}^{-1}\gamma(u)\tau^{\deg(\gamma)}u^{-1}$. So it suffices to show that $\phi(\rho_\infty^{\mathfrak{a}}(\sigma))$ equals

$$\phi(\rho_\infty^{\mathfrak{ab}}(\sigma\gamma)\rho_\infty^{\mathfrak{b}}(\gamma)^{-1}) = \phi_{\mathfrak{ab}}^{-1}\,\sigma\gamma(u)\tau^{\deg(\sigma\gamma)}u^{-1} \cdot u\tau^{-\deg(\gamma)}\gamma(u)^{-1}\phi_{\mathfrak{b}}$$
$$= \phi_{\mathfrak{ab}}^{-1}\sigma(\gamma(u))\tau^{\deg(\sigma)}\gamma(u)^{-1}\phi_{\mathfrak{b}}$$
$$= \phi_{\mathfrak{ab}}^{-1}\sigma(\phi_{\mathfrak{b}})\phi_{\mathfrak{a}} \cdot \phi_{\mathfrak{a}}^{-1}\sigma(\phi_{\mathfrak{b}}^{-1}\gamma(u))\tau^{\deg(\sigma)}(\phi_{\mathfrak{b}}^{-1}\gamma(u))^{-1}.$$

We showed in §4.1 that $\phi_{\mathfrak{ab}} = \sigma(\phi_{\mathfrak{b}})\phi_{\mathfrak{a}}$, so we need only prove that $\phi_{\mathfrak{a}}^{-1}\sigma(\phi_{\mathfrak{b}}^{-1}\gamma(u))\tau^{\deg(\sigma)}(\phi_{\mathfrak{b}}^{-1}\gamma(u))^{-1}$ and $\phi(\rho_\infty^{\mathfrak{a}}(\sigma))$ agree. By Lemma 3.2(ii), it thus suffices to show that $(\phi_{\mathfrak{b}}^{-1}\gamma(u))^{-1}\phi(F_\infty)\phi_{\mathfrak{b}}^{-1}\gamma(u) \subseteq \bar{k}((\tau^{-1}))$.

Take any $x \in F_\infty$. Since $u^{-1}\phi_x u$ belongs to $\bar{k}((\tau^{-1}))$, so does $\gamma(u^{-1}\phi_x u)$. By our choice of $\mathfrak{b}$, we have

$$\gamma(u^{-1}\phi_x u) = \gamma(u)^{-1}\gamma(\phi)_x\gamma(u) = \gamma(u)^{-1}(\mathfrak{b}*\phi)_x\gamma(u) = \gamma(u)^{-1}\phi_{\mathfrak{b}}\phi_x\phi_{\mathfrak{b}}^{-1}\gamma(u),$$

and hence $(\phi_{\mathfrak{b}}^{-1}\gamma(u))^{-1}\phi_x(\phi_{\mathfrak{b}}^{-1}\gamma(u))$ is an element of $\bar{k}((\tau^{-1}))$.

(ii) First note that

$$\phi(\rho_\infty^{\mathfrak{a}}(\sigma)\rho_\infty^{\mathfrak{b}}(\sigma)^{-1}) = \phi_{\mathfrak{a}}^{-1}\sigma(u)\tau^{\deg\sigma}u^{-1} \cdot u\tau^{-\deg\sigma}\sigma(u)^{-1}\phi_{\mathfrak{b}} = \phi_{\mathfrak{a}}^{-1}\phi_{\mathfrak{b}}.$$

In §4.1, we showed that $\phi_{\mathfrak{a}}^{-1}\phi_{\mathfrak{b}} = \phi_{w_1}\phi_{w_2}^{-1}$ where w_1 and w_2 belong to A and $w = w_1/w_2$ is the unique generator of $\mathfrak{ba}^{-1}$ satisfying $\varepsilon(w) = 1$. Therefore, $\rho_\infty^{\mathfrak{a}}(\sigma)\rho_\infty^{\mathfrak{b}}(\sigma)^{-1} = w$ as desired.

4.4. Proof of Lemma 3.4.
The map ρ_∞ is a homomorphism by Lemma 3.3(i). Fix a series $u = \sum_{i\geq 0}c_i\tau^{-i}$ as in Lemma 3.2(i). For $\sigma \in W_{H_A^+}$, we have $\mathrm{ord}_\infty(\rho_\infty(\sigma)) = d_\infty\,\mathrm{ord}_{\tau^{-1}}(\phi(\rho_\infty(\sigma))) = -d_\infty\deg(\sigma)$.

So to prove that ρ_∞ is continuous, we need only show that $\mathrm{Gal}(F^{\mathrm{sep}}/H_A^+\bar{k}) \xrightarrow{\rho_\infty} \mathcal{O}_\infty^\times$ is continuous. It suffices to show that for each $e \geq 1$, the homomorphism

$$\beta_e\colon\ \mathrm{Gal}(F^{\mathrm{sep}}/H_A^+\bar{k}) \xrightarrow{\rho_\infty} \mathcal{O}_\infty^\times \to (\mathcal{O}_\infty/\mathfrak{m}_\infty^e)^\times$$

has open kernel. For each $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/H_A^+\bar{k})$, we have $\phi(\rho_\infty(\sigma)) = \sigma(u)u^{-1}$. One can check that $\beta_e(\sigma) = 1$, equivalently $\mathrm{ord}_\infty(\rho_\infty(\sigma) - 1) \geq e$, if and only if $\mathrm{ord}_{\tau^{-1}}(\sigma(u)u^{-1} - 1) = \mathrm{ord}_{\tau^{-1}}(\sigma(u) - u)$ is at least ed_∞. Thus the kernel of β_e is $\mathrm{Gal}(F^{\mathrm{sep}}/L_e)$ where L_e is the finite extension of $H_A^+\bar{k}$ generated by the set $\{c_i\}_{0\leq i<ed_\infty}$

It remains to prove that ρ_∞ is unramified at all places of H_A^+ not lying over ∞. Let L' be the subfield of F^{sep} fixed by $\ker(\rho_\infty)$; it is the extension of $H_A^+\bar{k}$ generated by the set $\{c_i\}_{i\geq 0}$. The field L' does not depend on the choice of u, since ρ_∞ does not. In Remark 4.1, we saw that the extension L of $F\bar{k}$ generated by the coefficients of a particular u was unramified at all places of F away from ∞. Therefore, L' is unramified at all places of F away from ∞, since L and H_A^+ both have this property.

4.5. **Proof of Lemma 3.6.** First consider the case $\lambda \neq \infty$. For each $e \geq 1$, we have $\mathrm{Frob}_\mathfrak{p}(\xi) = \phi_\mathfrak{p}(\xi)$ for all $\xi \in \phi[\lambda^e] \subseteq F^{\mathrm{sep}}$; this was observed by Hayes [Hay92, p.28], for a proof see [Gos96, §7.5]. Thus the map $V_\lambda(\phi_\mathfrak{p})\colon V_\lambda(\phi) \to V_\lambda(\phi)$ equals $V_\lambda(\mathrm{Frob}_\mathfrak{p})$, and hence $\rho_\lambda^\mathfrak{p}(\mathrm{Frob}_\mathfrak{p}) = V_\lambda(\phi_\mathfrak{p})^{-1} \circ V_\lambda(\mathrm{Frob}_\mathfrak{p}) = 1$.

We may now assume that $\lambda = \infty$. Let $\overline{F}_\mathfrak{p}$ be an algebraic closure of $F_\mathfrak{p}$. The field $\overline{F}_\mathfrak{p}$ has a unique place extending the $\mathfrak{p}$-adic place of $F_\mathfrak{p}$ and let $\overline{\mathcal{O}}_\mathfrak{p} \subseteq \overline{F}_\mathfrak{p}$ be the corresponding valuation ring. The residue field of $\overline{\mathcal{O}}_\mathfrak{p}$ is an algebraic closure of $\mathbb{F}_\mathfrak{p}$ that we denote by $\overline{\mathbb{F}}_\mathfrak{p}$, and let $r_\mathfrak{p}\colon \overline{\mathcal{O}}_\mathfrak{p} \to \overline{\mathbb{F}}_\mathfrak{p}$ be the reduction homomorphism.

Choose an embedding $\overline{F} \hookrightarrow \overline{F}_\mathfrak{p}$. The restriction map $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p}) \to \mathrm{Gal}(\overline{F}/F)$ is an inclusion that is well defined up to conjugation. So after conjugating, we may assume that $\mathrm{Frob}_\mathfrak{p}$ lies in $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$; it thus acts on $\overline{\mathcal{O}}_\mathfrak{p}$ and we have $r_\mathfrak{p}(\mathrm{Frob}_\mathfrak{p}(\xi)) = r_\mathfrak{p}(\xi)^{N(\mathfrak{p})}$ for all $\xi \in \overline{\mathcal{O}}_\mathfrak{p}$. We will also denote by $r_\mathfrak{p}$ the map $\overline{\mathcal{O}}_\mathfrak{p}((\tau^{-1})) \to \overline{\mathbb{F}}_\mathfrak{p}((\tau^{-1}))$ where one reduces the coefficients of the series.

In §2.4, we noted that the coefficients of ϕ_x are integral over A for each $x \in A$. This implies that $\phi(A) \subseteq \overline{\mathcal{O}}_\mathfrak{p}[\tau]$. Define the homomorphism

$$\bar{\phi}\colon A \xrightarrow{\phi} \overline{\mathcal{O}}_\mathfrak{p}[\tau] \xrightarrow{r_\mathfrak{p}} \overline{\mathbb{F}}_\mathfrak{p}[\tau].$$

The map $\bar{\phi}$ is a Drinfeld module over $\overline{\mathbb{F}}_\mathfrak{p}$ of rank 1 since ϕ is normalized. Since ϕ is normalized, we find that $\phi_x \in \overline{\mathcal{O}}_\mathfrak{p}((\tau^{-1}))^\times$ for all non-zero $x \in A$. Therefore, $\phi(F) \subseteq \overline{\mathcal{O}}_\mathfrak{p}((\tau^{-1}))$. Using that $\overline{\mathcal{O}}_\mathfrak{p}((\tau^{-1}))$ is complete with respect to $\mathrm{ord}_{\tau^{-1}}$ and that (2.1) holds with $r = 1$, we deduce that $\phi(F_\infty) \subseteq \overline{\mathcal{O}}_\mathfrak{p}((\tau^{-1}))$.

We claim that $r_\mathfrak{p}$ induces an isomorphism $\phi(F_\infty) \to \bar{\phi}(F_\infty)$. Since $\bar{\phi}$ is a Drinfeld module, and hence injective, the map $r_\mathfrak{p}$ induces an isomorphism $\phi(A) \to \bar{\phi}(A)$ which then extends to an isomorphism $\phi(F) \to \bar{\phi}(F)$ of their quotient fields. The map $r_\mathfrak{p}\colon \phi(F) \to \bar{\phi}(F)$ preserves the valuation $\mathrm{ord}_{\tau^{-1}}$ so by the uniqueness of completions, $r_\mathfrak{p}$ also gives an isomorphism $\phi(F_\infty) \to \bar{\phi}(F_\infty)$.

Thus to prove that $\rho_\infty^\mathfrak{p}(\mathrm{Frob}_\mathfrak{p})$ equals 1, it suffices to show that $r_\mathfrak{p}(\phi(\rho_\infty^\mathfrak{p}(\mathrm{Frob}_\mathfrak{p}))) = 1$. Lemma 3.2(i), along with Remark 4.1, shows that there is a series $u \in \overline{\mathcal{O}}_\mathfrak{p}[\![\tau^{-1}]\!]^\times$ with coefficients in F^{sep} such that $u^{-1}\phi(F_\infty)u \subseteq \bar{k}((\tau^{-1}))$. With such a series u, we have

$$\phi(\rho_\infty(\mathrm{Frob}_\mathfrak{p})) = \phi_\mathfrak{p}^{-1}\,\mathrm{Frob}_\mathfrak{p}(u)\tau^d u^{-1}$$

where $d = \deg(\mathrm{Frob}_\mathfrak{p})$. The polynomial $\phi_\mathfrak{p} \in F[\tau]$ has coefficients in $\mathcal{O}_\mathfrak{p}$ and $r_\mathfrak{p}(\phi_\mathfrak{p}) = \tau^d$; thus $\phi_\mathfrak{p}^{-1}$ belongs to $\overline{\mathcal{O}}_\mathfrak{p}((\tau^{-1}))$ and $r_\mathfrak{p}(\phi_\mathfrak{p}^{-1}) = \tau^{-d}$. We have $r_\mathfrak{p}(\mathrm{Frob}_\mathfrak{p}(u)) = \tau^d r_\mathfrak{p}(u)\tau^{-d}$. Therefore,

$$r_\mathfrak{p}(\phi(\rho_\infty(\mathrm{Frob}_\mathfrak{p}))) = r_\mathfrak{p}(\phi_\mathfrak{p}^{-1})r_\mathfrak{p}(\mathrm{Frob}_\mathfrak{p}(u))\tau^d r_\mathfrak{p}(u)^{-1} = \tau^{-d}\cdot\tau^d r_\mathfrak{p}(u)\tau^{-d}\cdot\tau^d r_\mathfrak{p}(u)^{-1} = 1.$$

5. The rational function field

We return to the rational function field $F = k(t)$ where k is a finite field with q elements. Using our constructions, we shall recover the description of F^{ab} given by Hayes in [Hay74]; he expressed F^{ab} as the compositum of three linearly disjoint fields over F. In particular, we will explain how two of these fields arise naturally from our representation ρ_∞.

We define $A = k[t]$; it is the subring of F consisting of functions that are regular away from a unique place ∞ of F. We have $\mathbb{F}_\infty = k$, $F_\infty = k((t^{-1}))$ and $\mathrm{ord}_\infty\colon F_\infty^\times \to \mathbb{Z}$ is the valuation for

which $\mathrm{ord}_\infty(t^{-1}) = 1$. Let $\varepsilon\colon F_\infty^\times \to k^\times$ be the unique sign function of F_∞ that satisfies $\varepsilon(t^{-1}) = 1$. Those $x \in F_\infty^\times$ for which $\varepsilon(x) = 1$ form the subgroup $F_\infty^+ := \langle t\rangle(1 + t^{-1}k[\![t^{-1}]\!]) = \langle t\rangle(1 + \mathfrak{m}_\infty)$.

Recall that the Carlitz module is the homomorphism $\phi\colon A \to F[\tau]$, $a \mapsto \phi_a$ of k-algebras for which $\phi_t = t + \tau$. In the notation of §2.4, ϕ is a Hayes module for ε. The coefficients of ϕ_t lie in F, so the normalizing field H_A^+ for (F, ∞, ε) equals F. We saw that $\mathrm{Gal}(H_A^+/F)$ acts transitively on X_ε, so $X_\varepsilon = \{\phi\}$.

For each place λ of F (including ∞!), we have defined a continuous homomorphism $\rho_\lambda\colon W_F^{\mathrm{ab}} \to F_\lambda^\times$, $\sigma \mapsto \rho_\lambda^A(\sigma)$. The representation ρ_λ is characterized by the property that

$$\rho_\lambda(\mathrm{Frob}_{\mathfrak{p}}) = \mathfrak{p}$$

holds for each monic irreducible polynomial $\mathfrak{p} \in A = k[t]$ not corresponding to λ (combine Lemmas 3.1(ii), 3.3(ii) and 3.6 to show that $\rho_\lambda^A(\mathrm{Frob}_{\mathfrak{p}})\rho_\lambda^{(\mathfrak{p})}(\mathrm{Frob}_{\mathfrak{p}})^{-1} = \rho_\lambda^A(\mathrm{Frob}_{\mathfrak{p}})$ is the unique generator w of $\mathfrak{p}A$ that satisfies $\varepsilon(w) = 1$). For $\lambda \neq \infty$, the representation ρ_λ has image in $\mathcal{O}_\lambda^\times$, so it extends to a continuous representation $\mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathcal{O}_\lambda^\times$. The image of ρ_∞ lies in F_∞^+, but is unbounded (so it does not extend to a Galois representation).

Combining the representations ρ_λ together, we obtain a continuous homomorphism

$$(5.1) \qquad \prod_\lambda \rho_\lambda\colon W_F^{\mathrm{ab}} \to F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times.$$

By composing $\prod_\lambda \rho_\lambda$ with the quotient map $\mathbb{A}_F^\times \to C_F$, we obtain a continuous homomorphism $\rho\colon W_F^{\mathrm{ab}} \to C_F$. Corollary 3.7 says that ρ is an isomorphism of topological groups (and that the inverse of $\sigma \mapsto \rho(\sigma)^{-1}$ is the Artin map θ_F). The quotient map $F_\infty^+ \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times \to C_F$ is actually an isomorphism, so the map (5.1) is also an isomorphism. Taking profinite completions, we obtain an isomorphism

$$(5.2) \qquad \mathrm{Gal}(F^{\mathrm{ab}}/F) \xrightarrow{\sim} \widehat{F_\infty^+} \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times = \widehat{\langle t\rangle} \cdot (1 + \mathfrak{m}_\infty) \times \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times.$$

Using this isomorphism, we can now describe three linearly disjoint abelian extensions of F whose compositum is F^{ab}.

5.0.1. *Torsion points.* The representation $\chi := \prod_{\lambda \neq \infty} \rho\colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \prod_{\lambda \neq \infty} \mathcal{O}_\lambda^\times = \widehat{A}^\times$ arises from the Galois action on the torsion points of ϕ as described in §1.4. The fixed field in F^{ab} of $\ker(\chi)$ is the field $K_\infty := \cup_m F(\phi[m])$ where the union is over all monic polynomials m of A, and we have an isomorphism $\mathrm{Gal}(K_\infty/F) \xrightarrow{\sim} \widehat{A}^\times$. The field K_∞, which was first given be Carlitz, is a geometric extension of F that is tamely ramified at ∞.

5.0.2. *Extension of constants.* Define the homomorphism $\deg\colon W_F^{\mathrm{ab}} \to \mathbb{Z}$ by $\sigma \mapsto -\mathrm{ord}_\infty(\rho_\infty(\sigma))$; this agrees with our usual definition of $\deg(\sigma)$ [it is easy to show that $\mathrm{ord}_{\tau^{-1}}(\phi(\rho_\infty(\sigma)))$ equals $\mathrm{ord}_{\tau^{-1}}(\tau^{\deg(\sigma)}) = -\deg(\sigma)$, and then use (2.1) with $r = d_\infty = 1$]. The map $\deg$ thus factors through $W(\bar{k}(t)/k(t)) \xrightarrow{\sim} \mathbb{Z}$, where $W(\bar{k}(t)/k(t))$ is the group of $\sigma \in \mathrm{Gal}(\bar{k}(t)/k(t))$ that act on $\bar{k}$ as an integral power of q. Of course, $\bar{k}(t)/k(t)$ is an abelian extension with $\mathrm{Gal}(\bar{k}(t)/k(t)) \xrightarrow{\sim} \widehat{\mathbb{Z}}$.

5.0.3. *Wildly ramified extension.* Define the homomorphism

$$W_F^{\mathrm{ab}} \to 1 + \mathfrak{m}_\infty, \quad \sigma \mapsto \rho_\infty(\sigma)/t^{\deg(\sigma)};$$

it is well defined since $\mathrm{ord}_\infty(\rho_\infty(\sigma)) = -\deg(\sigma)$ and since the image of ρ_∞ is contained in $F_\infty^+ = \langle t\rangle(1 + \mathfrak{m}_\infty)$. Since $1 + \mathfrak{m}_\infty$ is compact, this gives rise to a Galois representation

$$\beta\colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to 1 + \mathfrak{m}_\infty.$$

Let L_∞ be the fixed field of $\ker(\beta)$ in F^{ab}. The field L_∞/F is an abelian extension of F that is unramified away from ∞ and is wildly ramified at ∞; it is also a geometric extension of F.

We will now give an explicit description of this field. Define $a_0 = 1$ and for $i \geq 1$, we recursively choose $a_i \in F^{\mathrm{sep}}$ that satisfy the equation

$$(5.3) \qquad a_i^q - a_i = -ta_{i-1}.$$

This gives rise to a chain of field extensions, $F \subset F(a_1) \subset F(a_2) \subset F(a_3) \subset \cdots$. We claim that $L_\infty = \bigcup_i F(a_i)$.

The construction of ρ_∞ starts by finding an appropriate series $u \in \overline{F}[\![\tau^{-1}]\!]^\times$ with coefficients in F^{sep}. Define $u := \sum_{i \geq 0} a_i \tau^{-i}$ with the a_i defined as above. The recursive equations that the a_i satisfy $\phi_t u = u\tau$ (just multiply them out and check!). As shown in §4.2, this implies that $u^{-1}\phi(F_\infty)u \subseteq \bar{k}(\!(\tau^{-1})\!)$. For $\sigma \in W_F^{\mathrm{ab}}$, $\rho_\infty(\sigma)$ is then the unique element of F_∞^+ for which $\phi(\rho_\infty(\sigma)) = \sigma(u)\tau^{\deg(\sigma)}u^{-1}$ where $\sigma(u)$ is obtained by letting σ act on the coefficients of u. In particular, $\phi(\beta(\sigma)) = \sigma(u)u^{-1}$ for all $\sigma \in \mathrm{Gal}(L_\infty/F)$ since L_∞/F is a geometric extension. We find that $\beta(\sigma) = 1$ if and only if $\sigma(u) = u$, and thus L_∞ is the extension of F generated by the set $\{a_i\}_{i \geq 1}$.

Using the isomorphism (5.2), we find that F^{ab} is the compositum of the fields K_∞, $\bar{k}(t)$ and L_∞, and that they are linearly disjoint over F. This is exactly the description given by Hayes in [Hay74, §5]; the advantage of our representation ρ_∞ is that the fields $\bar{k}(t)$ and L_∞ arise naturally.

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